THE EXPONENTIAL AND THE LOGARITHM

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In this note, we will introduce the exponential function $\exp\colon \mathbb{R} \to (0,\infty)$ via a power series, and study the natural logarithm, which we define to be the inverse function to exp.

To begin with, we recall that the radius of convergence of the power series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

is infinite. Hence the series converges to a function on $\mathbb{R}$, allowing us to make the following definition:

Definition 1. For every $x \in \mathbb{R}$, we *define* $\exp(x)$ to be

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

We have the following properties of the exponential function:

Proposition 1. *(a)* exp *is an infinitely differentiable function on* $\mathbb{R}$, *and*

$$\frac{d}{dx}\exp(x) = \exp(x).$$

(b) $\exp(0) = 1$, *and* $\exp(x+y) = \exp(x)\exp(y)$ *for all* $x, y \in \mathbb{R}$. *In particular,*

$$\exp(-x) = \frac{1}{\exp(x)} \quad \textit{for all } x \in \mathbb{R}. \tag{1}$$

(c) $\exp(x) > 0$ *for all* $x \in \mathbb{R}$, *and* $\exp\colon \mathbb{R} \to (0,\infty)$ *is strictly increasing on* $\mathbb{R}$.

(d) $\exp(x) \geq 1 + x$ *for all* $x \in \mathbb{R}$; *in particular,* $\exp(x) \to +\infty$ *as* $x \to +\infty$, *and* $\exp(x) \to 0$ *as* $x \to -\infty$.

(e) $\exp\colon \mathbb{R} \to (0,\infty)$ *is bijective.*

(f) For any $x \in \mathbb{R}$, *and any* $m \in \mathbb{Z}$, $n \in \mathbb{N}$, *we have*

$$\exp\left(\frac{mx}{n}\right) = [\exp(x)]^{\frac{m}{n}}.$$

In particular, the last part of the proposition says the function $\exp\colon \mathbb{R} \to (0,\infty)$ has an inverse. We will *define* the natural logarithm to be this inverse, and write the natural logarithm as $\log\colon (0,\infty) \to \mathbb{R}$.

Proof. (a) The radius of convergence of the power series defining exp is infinite. As a result, exp is infinitely differentiable on $\mathbb{R}$, and

$$\frac{d}{dx}\exp(x) = \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{x^k}{k!}\right) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \exp(x).$$

(b) First,

$$\exp(0) = 1 + \sum_{k=1}^{\infty} \frac{0^k}{k!} = 1.$$

Also, let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \exp(-x)\exp(x).$$

Then $f(0) = \exp(0)\exp(0) = 1$. Also, by chain and product rule, we have that

$$f'(x) = -\exp(-x)\exp(x) + \exp(-x)\exp(x) = 0.$$

This shows that f is constant equal to 1, i.e.

$$1 = f(x) = \exp(-x)\exp(x)$$

for all $x \in \mathbb{R}$. Hence (1) holds, as desired.

Finally, for each fixed $y \in \mathbb{R}$, let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \exp(-x)\exp(x+y) - \exp(y).$$

Then $g(0) = \exp(0)\exp(y) - \exp(y) = 0$. Also, by chain and product rule, we have that

$$g'(x) = -\exp(-x)\exp(x+y) + \exp(-x)\exp(x+y) = 0.$$

This shows that g is constant equal to 0, i.e.

$$\exp(-x)\exp(x+y) = \exp(y) \quad \text{for all } x \in \mathbb{R}.$$

Multiplying both sides by $\exp(x)$, and using (1), we obtain $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$, as desired.

(c) When $x \geq 0$, it is clear from the definition of $\exp(x)$ that

$$\exp(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} \geq 1.$$

When $x \leq 0$, we then have (by (1)) that

$$\exp(x) = \frac{1}{\exp(-x)} \in (0, 1].$$

Hence $\exp(x) > 0$ for all $x \in \mathbb{R}$. In particular,

$$\frac{d}{dx}\exp(x) = \exp(x) > 0,$$

and $\exp\colon \mathbb{R} \to (0,\infty)$ is strictly increasing on $\mathbb{R}$.

(d) Let $h(x) = \exp(x) - (1+x)$. Then $h(0) = \exp(0) - 1 = 0$, and

$$h'(x) = \exp(x) - 1 \begin{cases} \geq 0 & \text{if } x \geq 0 \\ \leq 0 & \text{if } x \leq 0. \end{cases}$$

In particular, $h(x) \geq h(0) = 0$ for all $x \in \mathbb{R}$, i.e. $\exp(x) \geq 1 + x$ for all $x \in \mathbb{R}$.

This shows $\exp(x) \to +\infty$ as $x \to +\infty$, and $\exp(x) = 1/\exp(-x) \to 0$ as $x \to -\infty$.

(e) Since $\exp\colon \mathbb{R} \to (0,\infty)$ is strictly increasing, in particular it is injective. Also, recall that exp is differentiable. In particular exp is continuous. Since

$$\lim_{x\to+\infty} \exp(x) = +\infty, \qquad \lim_{x\to-\infty} \exp(x) = 0,$$

by intermediate value theorem, we see that $\exp\colon \mathbb{R} \to (0,\infty)$ is surjective. This completes our proof.

(f) Part (b) above says $\exp(x+y) = \exp(x+y)$ for any $x, y \in \mathbb{R}$. In particular,

$$\exp(mx) = [\exp(x)]^m \quad \text{whenever } x \in \mathbb{R} \text{ and } m \in \mathbb{N}.$$

This certainly also holds when $m = 0$, since $\exp(0) = 1$. Also, if $x \in \mathbb{R}$ and m is a negative integer, then $\exp(mx) = 1/\exp(-mx) = 1/[\exp(-x)]^m = [\exp(x)]^m$ as well. Altogether, this shows

$$\exp(mx) = [\exp(x)]^m \quad \text{whenever } x \in \mathbb{R} \text{ and } m \in \mathbb{Z}. \tag{2}$$

Now for any $x \in \mathbb{R}$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have (by repeated applications of (2))

$$\left[\exp\left(\frac{mx}{n}\right)\right]^n = \exp\left(\frac{mx}{n} \cdot n\right) = \exp(mx) = [\exp(x)]^m.$$

Taking the n-th root on both sides, we yield the desired identity.

□

Let us *define* now e by

$$e = \exp(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots.$$

Then part (f) above implies that

$$\exp\left(\frac{m}{n}\right) = e^{\frac{m}{n}}$$

whenever $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Hence the exponential function coincides with the power of the number e if the power is a rational number; this motivates one to *define* e^x as $\exp(x)$, even when x is just real but not necessarily rational.

Next, we *define* the natural logarithm $\log\colon (0,\infty) \to \mathbb{R}$ to be the inverse function to $\exp\colon \mathbb{R} \to (0,\infty)$.

It is then rather easy to translate the properties of exp above, into properties of log:

Proposition 2. *(a)* $\log\colon (0,\infty) \to \mathbb{R}$ *is strictly increasing and bijective.*
(b) $\log$ *is differentiable on* $(0,\infty)$*, and*

$$\frac{d}{dx}\log x = \frac{1}{x}.$$

(c) $\log(1) = 0$*, and* $\log(xy) = \log(x) + \log(y)$ *for any* $x, y > 0$*. In particular,*

$$\log\left(\frac{1}{x}\right) = -\log(x) \quad \textit{for all } x > 0.$$

(d) For any $x \in \mathbb{R}$*, and any* $m \in \mathbb{Z}$*,* $n \in \mathbb{N}$*, we have*

$$\log\left(x^{\frac{m}{n}}\right) = \frac{m}{n}\log(x).$$

Proof. All these are obvious except (b). But (b) follows from the inverse function theorem. □

We can now derive the power series expansion for $\log(1+x)$. Note that this holds only when $|x| < 1$.

Proposition 3. *For* $|x| < 1$*, we have*

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots$$

Proof. In fact, the power series on the right hand side has coefficients

$$a_n = \frac{(-1)^{n-1}}{n}$$

so it radius of convergence is

$$\frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{\lim_{n\to\infty}\frac{n}{n+1}} = 1.$$

This allows us to define a function $f\colon (-1,1) \to \mathbb{R}$, by

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots \qquad \text{for } |x| < 1.$$

We can compute the derivative of $f(x)$ by the above theorem:

$$f'(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots \qquad \text{for } |x| < 1.$$

One can sum the right hand side above, since it is a geometric series:

$$f'(x) = \frac{1}{1+x} \qquad \text{for } |x| < 1.$$

Hence

$$f'(x) - \frac{d}{dx}\log(1+x) = 0 \qquad \text{for } |x| < 1.$$

From the mean-value theorem, it follows that $f(x) - \log(1+x)$ is a constant on $(-1, 1)$, i.e.

$$f(x) - \log(1+x) = f(0) - \log(1+0) = 0, \qquad \text{for } |x| < 1.$$

This proves the desired identity. □

Finally, we can now prove the following formula for $\exp(x)$:

Proposition 4. *For any $x \in \mathbb{R}$, the limit*

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

exists, and is equal to $\exp(x)$.

In particular, the limit $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ exists, and is equal to e.

Proof. Note that by part (d) of Proposition 2, whenever $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\left(1 + \frac{x}{n}\right)^n = \exp\left[n \log\left(1 + \frac{x}{n}\right)\right].$$

Now

$$\lim_{t\to+\infty} t \log\left(1 + \frac{x}{t}\right)$$

exists and is equal to x; this is just a simple application of L'Hopital's rule (which is justified since we know the derivative of log now). Hence

$$\lim_{n\to\infty} n \log\left(1 + \frac{x}{n}\right)$$

also exists and is equal to x. It follows (by continuity of exp) that

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} \exp\left[n \log\left(1 + \frac{x}{n}\right)\right]$$

also exists, and is equal to

$$\exp\left[\lim_{n\to\infty} n \log\left(1 + \frac{x}{n}\right)\right] = \exp(x),$$

as was to be proved. □